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Inequalities for Polynomials with Two Equal Coefficients

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1.1. An expression of the form $\sum_{\nu=-n}^{n} c_\nu e^{i\nu\theta}$, where the c_ν's are arbitrary complex numbers will be referred to as a trigonometric polynomial of degree n. By a polynomial of degree n we will mean the finite sum $\sum_{\nu=0}^{n} a_\nu z^\nu$, where $a_\nu \in \mathbb{C}$ $(\nu = 0, 1, \ldots, n)$.

According to Bernstein's inequality if t is a trigonometric polynomial of degree n such that

$$|t(\theta)| \leqslant 1 \qquad \text{for} \quad \theta \in \mathbb{R} \tag{1}$$

then (for references see [6])

$$|t'(\theta)| \leqslant n \qquad \text{for} \quad \theta \in \mathbb{R}. \tag{2}$$

In (2), equality holds *if and only if*

$$t(\theta) = c_{-n} e^{-in\theta} + c_n e^{in\theta}, \qquad |c_{-n}| + |c_n| = 1.$$

It was shown by van der Corput and Schaake [1] that in the case when $t(\theta)$ is *real for real values of* θ the much stronger conclusion

$$|t'(\theta) \pm int(\theta)| = \sqrt{\{t'(\theta)\}^2 + n^2\{t(\theta)\}^2} \leqslant n \tag{3}$$

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holds for all $\theta \in \mathbb{R}$. Inequality (3) is sharp for each θ; in fact, all real trigonometric polynomials of the form

$$t(\theta) = c_{-n} e^{-in\theta} + c_n e^{in\theta} \qquad (c_{-n} = \bar{c}_n, |c_n| = \tfrac{1}{2})$$

are extremal. The example $t(\theta) = e^{\pm in\theta}$ shows that for an arbitrary trigonometric polynomial of degree n the quantity $|t'(\theta) \pm int(\theta)|$ can be as large as $2n$, which is trivially its upper bound.

If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n such that

$$|p(z)| \leqslant 1 \qquad \text{for} \quad |z| = 1 \tag{4}$$

then $p(e^{i\theta}) = t(e^{i\theta})$, where t is a trigonometric polynomial of degree n satisfying (1) and so

$$|p'(z)| \leqslant n \qquad \text{for} \quad |z| = 1. \tag{5}$$

Here, equality holds if and only if

$$p(z) = a_n z^n \qquad (|a_n| = 1).$$

If $z^n \overline{p(1/\bar{z})} \equiv p(z)$, i.e., $a_k = \bar{a}_{n-k}$ for $0 \leqslant k \leqslant n$, then (for references see [6]) the right-hand side of (5) may be replaced by $n/2$. The question as to what happens if

$$z^n p(1/z) \equiv p(z) \text{ (i.e., } a_k = a_{n-k}) \qquad \text{for} \quad 0 \leqslant k \leqslant n \tag{6}$$

was taken up by Govil, Jain and Labelle [5] but remains unresolved. In [4] we showed that there exists a polynomial of degree n ($\geqslant 2$), namely

$$p(z) = \{(1 - iz)^2 + z^{n-2}(z - i)^2\}/4, \tag{7}$$

satisfying (6) for which

$$\max_{|z|=1} |p'(z)| \geqslant |p'(-i)| = n - 1 \geqslant (n-1) \max_{|z|=1} |p(z)|. \tag{8}$$

This is surprising since (6) is in some sense quite restrictive. It is clear that for a polynomial p satisfying (4) and (6) the sharp upper bound for $|p'(e^{i\theta})|$ would depend not only on n but also on θ. We shall see that for such polynomials

$$|p'(e^{2k\pi i/n})| \leqslant n - 1, \qquad k = 0, 1, \ldots, n-1, \tag{9}$$

and so the polynomial in (7) happens to be extremal for $\theta = -i$ if $n = 4, 8, 12, \ldots$. This remains true even if (6) is replaced by the much weaker assumption $a_0 = a_n$. In fact, we prove

THEOREM 1. *Let* $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial of degree* n $(\geqslant 2)$ *satisfying* (4). *If* $a_0 = a_n$, *then*

$$|p'(e^{i\theta})| \leqslant (n-1) + |a_0|\, |e^{in\theta}-1| \qquad \textit{for} \quad \theta \in \mathbb{R}, \tag{10}$$

and so in particular

$$|p'(e^{i\theta})| \leqslant n-1 \qquad \text{if} \quad e^{in\theta}=1. \tag{10'}$$

Remark. The example

$$p_\omega(z) = \{\omega^{n-2}(\omega - z)^2 + z^{n-2}(\omega+z)^2\}/4, \qquad \omega^n = 1,$$

shows that in (10′) equality can hold at any of the n-th roots of unity for *all* $n \geqslant 2$.

As a global upper bound for $|p'(e^{i\theta})|$, inequality (10) gives us only the trivial value n. But we will show how it can be used to obtain:

THEOREM 2. *Under the conditions of Theorem* 1 *we have*

$$|p'(z)| \leqslant n - \frac{1}{2} + \frac{1}{2(n+1)} \qquad \textit{for} \quad |z|=1. \tag{11}$$

If t is a trigonometric polynomial of degree n then

$$e^{in\theta}t(\theta) = p_1(e^{i\theta}), \qquad e^{-in\theta}t(\theta) = p_2(e^{-i\theta})$$

where p_1 and p_2 are polynomials of degree $2n$. Thus Theorems 1 and 2 readily imply:

COROLLARY 1. *Let* $t(\theta) = \sum_{\nu=-n}^{n} c_\nu e^{i\nu\theta}$ *be a trigonometric polynomial of degree* n *satisfying* (1). *If* $c_{-n} = c_n$ *(which is the case if for example* t *is a cosine polynomial), then*

$$|t'(\theta) \pm int(\theta)| \leqslant 2n - 1 + 2\,|c_n|\, |\sin n\theta| \qquad \textit{for} \quad \theta \in \mathbb{R}, \tag{12}$$

and so in particular

$$|t'(k\pi/n) \pm int(k\pi/n)| \leqslant 2n-1, \qquad k = 0, 1, \ldots, 2n-1. \tag{13}$$

Further

$$|t'(\theta) \pm int(\theta)| \leqslant 2n - \frac{1}{2} + \frac{1}{2(2n+1)} \qquad \textit{for} \quad \theta \in \mathbb{R}. \tag{14}$$

It is easily seen that

$$|t'(k\pi/n) + int(k\pi/n)| = 2n-1$$

for the trigonometric polynomial

$$t(\theta) = t_{n,k}(\theta) = e^{-in\theta}\{(1 - e^{i(\theta - (k\pi/n))})^2 + e^{2i(k\pi/n)}\, e^{2i(n-1)\theta}(1 + e^{i(\theta - (k\pi/n))})^2\}/4$$

which satisfies (1) and for which $c_{-n} = c_n = \frac{1}{4}$. We have

$$|t'(k\pi/n) - int(k\pi/n)| = 2n - 1$$

for

$$t\colon \theta \mapsto \overline{t_{n,k}(\theta)}.$$

1.2. It was proved by Duffin and Schaeffer [3] that if f is an entire function of exponential type τ satisfying

$$|f(x)| \leqslant 1 \qquad \text{for} \quad x \in \mathbb{R} \tag{15}$$

and is real on the real axis, then

$$|f'(x) \pm i\tau f(x)| \leqslant \tau \qquad \text{for} \quad x \in \mathbb{R}. \tag{16}$$

This result generalizes inequality (3) of van der Corput and Schaake since a trigonometric polynomial $t(\theta) = \sum_{\nu=-n}^{n} c_\nu e^{i\nu\theta}$ is an entire function of exponential type n of the complex variable θ. A cosine polynomial being an *even* entire function of exponential type one might wonder if Corollary 1 admits an extension to such functions. It turns out that the best possible upper bound is the trivial bound 2τ. To see this let ε be an arbitrary positive number less than τ (there is nothing to prove in the case $\tau = 0$) and consider the even entire function

$$f_{\tau,\varepsilon}(z) = e^{-i\tau z}\{(1 - ie^{i\varepsilon z})^2 + e^{2i(\tau-\varepsilon)z}(e^{i\varepsilon z} - i)^2\}/4$$

which is of exponential type τ and for $x \in \mathbb{R}$

$$\begin{aligned} |f_{\tau,\varepsilon}(x)| &\leqslant \tfrac{1}{4}(|1 - ie^{i\varepsilon x}|^2 + |e^{i\varepsilon x} - i|^2) \\ &= \tfrac{1}{4}(|e^{i\varepsilon x} + i|^2 + |e^{i\varepsilon x} - i|^2) \\ &\leqslant 1. \end{aligned}$$

Further, it is easily checked that

$$\left| f'_{\tau,\varepsilon}\left(\frac{(4k-1)\pi}{2\varepsilon}\right) + i\tau f_{\tau,\varepsilon}\left(\frac{(4k-1)\pi}{2\varepsilon}\right)\right| > 2\tau - \varepsilon, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

We have

$$\left|f'\left(\frac{(4k-1)\pi}{2\varepsilon}\right)-i\tau f\left(\frac{(4k-1)\pi}{2\varepsilon}\right)\right|>2\tau-\varepsilon, \qquad k=0,\ \pm 1,\ \pm 2,...$$

for

$$f\colon z\mapsto \overline{f_{\tau,\varepsilon}(\bar{z})}.$$

1.3. If p is a polynomial of degree n such that

$$|p(x)|\leqslant 1 \qquad \text{for} \quad -1\leqslant x\leqslant 1 \tag{17}$$

then $p(\cos\theta)$ is a cosine polynomial t of degree n satisfying (1) and so as a special case of Corollary 1 we obtain

COROLLARY 2. *Let* $T_n(x)=\cos n\arccos x$ *be the* nth *Chebyshev polynomial of the first kind. If* $p(x)=\sum_{\nu=0}^{n}a_\nu x^\nu$ *is a polynomial of degree* n *satisfying* (17), *then*

$$|np(x)\pm i\sqrt{1-x^2}\,p'(x)| \leqslant 2n-1+\frac{1}{2^{n-1}}|a_n|\sqrt{1-(T_n(x))^2}, \qquad -1\leqslant x\leqslant 1, \tag{18}$$

and so in particular

$$\left|np\left(\cos\frac{k\pi}{n}\right)\pm i\sin\frac{k\pi}{n}\,p'\left(\cos\frac{k\pi}{n}\right)\right| \leqslant 2n-1, \qquad k=0,1,...,n-1. \tag{19}$$

Further

$$|np(x)\pm i\sqrt{1-x^2}\,p'(x)|\leqslant 2n-\frac{1}{2}+\frac{1}{4(n+1)}, \qquad -1\leqslant x\leqslant 1. \tag{20}$$

It is clear from the context that inequality (19) is sharp.

1.4. *A lower bound for* $\max_{|z|=1}|p'(z)|$.

Let $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ be a polynomial of degree n ($\geqslant 2$) such that $a_0=a_n$ and $\max_{|z|=1}|p(z)|=1$. The example $p(z)=z$ shows that for such a polynomial $\max_{|z|=1}|p'(z)|$ *may be as small as* 1. On the other hand, we have

THEOREM 3. *If* $p(z)=\sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree n such that* $|a_0|=|a_n|$ *and* $\max_{|z|=1}|p(z)|=1$, *then*

$$\max_{|z|=1}|p'(z)| \begin{cases} \geqslant 1+\dfrac{n-3}{n+1}|a_0| & \text{if } n\geqslant 3 \\ \geqslant 1 & \text{if } n=2. \end{cases}$$

2. AN INTERPOLATION FORMULA

For the proof of Theorem 1 we need the following

LEMMA 1. *If* $p(z)=\sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n $(\geqslant 3)$ *then for all real* γ *we have*

$$\begin{aligned} &a_0+((n-1)\,p(z)-zp'(z)+a_n z^n-2a_0)\,e^{i\gamma} \\ &\quad+(zp'(z)-p(z)-2a_n z^n+a_0)\,e^{2i\gamma}+a_n z^n\,e^{3i\gamma} \\ &=\frac{1}{n-2}\,e^{i\gamma}\sin^2(\gamma/2) \\ &\quad\times\sum_{k=1}^{n-2}\frac{e^{-(2k\pi+\gamma)i/(n-2)}}{\sin^2((2k\pi+\gamma)/2(n-2))}\,p(ze^{(2k\pi+\gamma)i/(n-2)}), \end{aligned} \tag{21}$$

with

$$\frac{1}{n-2}\sin^2(\gamma/2)\sum_{k=1}^{n-2}\frac{1}{\sin^2((2k\pi+\gamma)/2(n-2))}=n-2. \tag{22}$$

Proof. Let γ ($\not\equiv 0 \pmod{2\pi}$) be an arbitrary real number. Further, let z be any complex number and consider the integral

$$I(\rho)=\int_{|\zeta|=\rho}F(\zeta)\,d\zeta$$

where

$$F(\zeta)=\frac{p(\zeta)}{(\zeta-z)^2\zeta(\zeta^{n-2}-e^{i\gamma}z^{n-2})}.$$

Clearly

$$I(\rho)\to a_n \qquad \text{as} \quad \rho\to\infty, \tag{23}$$

whereas the residues of F at its poles z, 0 and $ze^{i(\gamma+2k\pi)/(n-2)}$, $k=1,\dots,n-2$, are

$$-\frac{1}{4}\frac{1}{z^n}\frac{e^{-i\gamma}}{\sin^2(\gamma/2)}\{(1-e^{i\gamma})\,zp'(z)-(1-e^{i\gamma})\,p(z)-(n-2)\,p(z)\},$$

$$-\frac{1}{z^n}e^{-i\gamma}a_0$$

and

$$-\frac{1}{4}\frac{1}{z^n}\frac{1}{n-2}e^{-i\gamma}\frac{e^{-(2k\pi+\gamma)i/(n-1)}}{\sin^2((2k\pi+\gamma)/2(n-2))}p(ze^{(2k\pi+\gamma)i/(n-2)}),\ k=1,\dots,n-2,$$

respectively. Hence by the theorem of residues

$$\begin{aligned}&4a_nz^n e^{i\gamma}\sin^2(\gamma/2)\\&\quad-\{(n-2)\,p(z)+(1-e^{i\gamma})\,p(z)-(1-e^{i\gamma})\,zp'(z)\}+4a_0\sin^2(\gamma/2)\\&\quad=-\frac{1}{n-2}\sin^2(\gamma/2)\\&\qquad\times\sum_{k=1}^{n-2}\frac{e^{-(2k\pi+\gamma)i/(n-2)}}{\sin^2((2k\pi+\gamma)/2(n-2))}p(ze^{(2k\pi+\gamma)i/(n-2)}),\end{aligned}$$

i.e.,

$$\begin{aligned}&(e^{3i\gamma}-2e^{2i\gamma}+e^{i\gamma})\,a_nz^n\\&\quad+\{(n-1)\,e^{i\gamma}p(z)-e^{2i\gamma}p(z)-(e^{i\gamma}-e^{2i\gamma})\,zp'(z)\}\\&\quad+(e^{2i\gamma}-2e^{i\gamma}+1)\,a_0\\&=\frac{1}{n-2}e^{i\gamma}\sin^2(\gamma/2)\\&\quad\times\sum_{k=1}^{n-2}\frac{e^{-(2k\pi+\gamma)i/(n-2)}}{\sin^2((2k\pi+\gamma)/2(n-2))}p(ze^{(2k\pi+\gamma)i/(n-2)})\end{aligned}$$

which is the same as (21). The assumption "$\gamma\not\equiv 0 \pmod{2\pi}$" can obviously be dropped. Formula (21) when applied to z^{n-1} (or to z) readily leads us to the identity (22).

3.1. *Proof of Theorem* 1. If $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n satisfying (4), then (21) in conjunction with (22) and a result of van der Corput and Visser [2] implies that

$$|a_0|+|zp'(z)-p(z)-2a_nz^n+a_0|\leqslant n-2,\qquad |z|=1,\ n\geqslant 3.$$

In the case when $a_0 = a_n$, this latter inequality can be written as

$$|a_n z^n| + |zp'(z) - p(z) - 2a_n z^n + a_0| \leqslant n-2, \qquad |z|=1, n \geqslant 3$$

from which we readily obtain (10) for $n \geqslant 3$.

In the case $n=2$, $p(z)$ has the form $a_0(z^2+1)+a_1 z$ so that

$$e^{-i\theta} p(e^{i\theta}) = 2a_0 \cos\theta + a_1.$$

Thus

$$|p'(e^{i\theta})| \leqslant |p(e^{i\theta})| + 2\,|a_0 \sin\theta|$$

which gives us the desired estimate.

Proof of Theorem 2. From (10) it readily follows that (11) holds provided $|a_0| \leqslant \frac{1}{4}((n+2/(n+1))$. In case $|a_0| > \frac{1}{4}((n+2)/(n+1))$ we may use the known estimate [7, p. 125]

$$|p'(e^{i\theta})| \leqslant n - \frac{2n}{n+2}\,|a_0|, \qquad \theta \in \mathbb{R},$$

to obtain the desired conclusion.

Proof of Theorem 3. Let

$$P(z) = p(z) - a_0 \qquad \text{and} \qquad Q(z) = z^n\,\overline{P(1/\bar{z})} = \bar{a}_1 z^{n-1} + \bar{a}_2 z^{n-2} + \cdots + \bar{a}_n.$$

From

$$Q(e^{i\theta}) = e^{in\theta}\,\overline{P(e^{i\theta})}, \qquad \theta \in \mathbb{R},$$

it follows that if $|P(e^{i\theta_0})| = \max_{|z|=1} |P(z)| = M$ (say), then

$$|P'(e^{i\theta_0})| \geqslant Mn - |Q'(e^{i\theta_0})|. \tag{24}$$

Further, since Q is a polynomial of degree $n-1$ such that $\max_{|z|=1} |Q(z)| = M$ and $|Q(0)| = |a_n| = |a_0|$, we have

$$|Q'(e^{i\theta})| \leqslant M(n-1) - \frac{2(n-1)}{n+1}\,|a_0|, \qquad \theta \in \mathbb{R}.$$

Thus, we obtain

$$\max_{|z|=1} |P'(z)| \geqslant |P'(e^{i\theta_0})| \geqslant M + \frac{2(n-1)}{n+1}\,|a_0|.$$

This gives us the desired result for $n \geqslant 3$ since $M \geqslant 1 - |a_0|$.

In the case $n=2$ we clearly have

$$\begin{aligned}\max_{|z|=1} |p'(z)| = \max_{|z|=1} |2a_2 z + a_1| &= 2\,|a_2| + |a_1| \\ &= |a_2| + |a_1| + |a_0| \\ &\geqslant \max_{|z|=1} |p(z)| \\ &= 1.\end{aligned}$$

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